Consider $a_{11} \ a_{12} \ \ldots \ a_{1m}$ row min $a_{1j}$

$a_{21} \ a_{22} \ \ldots \ a_{2m}$ $a_{2j}$

$I \ a_{31}$

$\vdots$

$a_{n1} \ a_{n2} \ \ldots \ a_{nm} \ a_{nj}$

Column max $a_{i1} \ a_{i2} \ \ldots \ a_{im}$

$V^+ =$ smallest max

$V^- =$ lower bound of the game. In the worst case, Player I is guaranteed
to get $V^-$. I.e., Player I's payoff floor

$V^+ =$ upper bound of the game. In the worst case, Player II is guaranteed
to lose at most $V^+$ amount. I.e., Player II's loss ceiling.

A game has a value iff $V^- = V^+$

$a_{ij} = V^- = V^+$ is the Saddle Point (i.e., optimal play) for pure
strategies.
Theorem: If one player of the game employs a fixed strategy, then the opponent has an optimal strategy that is pure.

Consider:

\[
\begin{array}{c|cc}
& 0 & 1 \\
\hline
1-p & a & b \\
p & c & d \\
\end{array}
\]

(a)

\[
\begin{array}{c|cc}
& 0 & 1 \\
\hline
1-p & a & b \\
p & c & d \\
\end{array}
\]

(b)

\[\text{EU}_a = (1-p) \times 1 \times a + p \times 1 \times c = (c-a)p + a\]
\[\text{EU}_b = (1-p) \times 1 \times b + p \times 1 \times d = (d-b)p + b\]

\[\text{ER}(p) = \text{the lesser of} \{\text{EU}_a, \text{EU}_b\}\]

For Penny matching game:

\[
\begin{array}{c|cc}
& H & T \\
\hline
H & 1 & -1 \\
T & -1 & 1 \\
\end{array}
\]

\[a = d = 1 \quad \text{EU}_a = 1 - 2p\]
\[b = c = -1 \quad \text{EU}_b = 2p - 1\]
Consider the game

\[ EV_a = (-1 - 3) P + 3 = -4P + 3 \]
\[ EV_b = (-2 - 1) P + 1 = -3P + 1 \]

Consider the game

\[ \begin{array}{cc}
0 & 1 \\
2 & 1 \\
\end{array} \]
\[ EV_a = 2P \]
\[ EV_b = 1 \]
Eliminating dominated strategies

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0,2</td>
<td>3,1</td>
<td>2,3</td>
</tr>
<tr>
<td>M</td>
<td>1,4</td>
<td>2,1</td>
<td>4,1</td>
</tr>
<tr>
<td>D</td>
<td>2,1</td>
<td>4,4</td>
<td>3,2</td>
</tr>
</tbody>
</table>

\[1\] \quad \text{M} \quad \begin{array}{c|c|c|c}
\text{L} & 1.4 & 2.1 & 4.1 \\
\text{D} & 2.1 & 4.4 & 3.2 \\
\end{array}

\[2\] \quad \text{D} \quad \begin{array}{c|c|c|c}
\text{L} & \text{C} & \text{D} \\
\text{M} & 1.4 & 2.1 & 4.1 \\
\text{D} & 2.1 & 4.4 & 3.2 \\
\end{array}

\[3\] \quad \text{D} \quad \begin{array}{c|c|c|c}
\text{L} & \text{C} & \text{D} \\
\text{M} & 1.4 & 2.1 & 4.1 \\
\text{D} & 2.1 & 4.4 & 3.2 \\
\end{array}

\[4\] \quad \text{D} \quad \begin{array}{c|c|c|c}
\text{L} & \text{C} & \text{D} \\
\text{M} & 1.4 & 2.1 & 4.1 \\
\text{D} & 2.1 & 4.4 & 3.2 \\
\end{array}

Backward induction

\[1\] \quad \text{W} \quad \begin{array}{c|c}
\text{LJ} & \text{LJ} \\
\text{W} & \text{C} \\
\end{array}

\[2\] \quad \text{W} \quad \begin{array}{c|c}
\text{LJ} & \text{LJ} \\
\text{W} & \text{C} \\
\end{array}

\[3\] \quad \text{W} \quad \begin{array}{c|c}
\text{LJ} & \text{LJ} \\
\text{W} & \text{C} \\
\end{array}

\[4\] \quad \text{W} \quad \begin{array}{c|c}
\text{LJ} & \text{LJ} \\
\text{W} & \text{C} \\
\end{array}

\[5\] \quad \text{W} \quad \begin{array}{c|c}
\text{LJ} & \text{LJ} \\
\text{W} & \text{C} \\
\end{array}

W - Wait  C - Climb  BJ - Bj  John  LJ - Lide John

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