A Perfect Information Game in Extensive Form 

\((N, A, H, Z, X, P, \sigma, u)\)

Players actions

Nonterminal nodes

Terminal nodes

\(X : H \rightarrow 2A\)

\(H \cap Z = \emptyset\)

Possible actions at each choice node

\(P : H \rightarrow N\) assign players to nonterminal nodes.

\(T : H \times A \rightarrow H \cup U \cup Z\) successor function

\(U = (u_1, \ldots, u_n)\) where \(u_i : Z \rightarrow R\) assigns utility for player \(i\) at a terminal node.

E.g. 1

The Sharing Game. I chooses to split 2 gifts, II chooses to accept or reject

\[
S_I = \{2-0, 1-1, \Phi-2\} \\
S_{II} = \{(Yes, Yes, Yes), (Yes, Yes, No), (Yes, No, Yes), (Yes, No, No), (No, Yes, Yes), (No, Yes, No), (No, No, Yes), (No, No, No)\}
\]

E.g. 2

\[
S_I = \{(A, G), (A, H), (B, G), (B, H)\} \\
S_{II} = \{(C, E), (C, F), (D, E), (D, F)\}
\]
<table>
<thead>
<tr>
<th></th>
<th>(C,E)</th>
<th>(C,F)</th>
<th>(D,E)</th>
<th>(D,F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A,G)</td>
<td>3,8</td>
<td>3,8*</td>
<td>8,3</td>
<td>8,3</td>
</tr>
<tr>
<td>(A,H)</td>
<td>3,8*</td>
<td>3,8*</td>
<td>8,3</td>
<td>8,3</td>
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<tr>
<td>(D,G)</td>
<td>5,5</td>
<td>2,10</td>
<td>5,5</td>
<td>2,10</td>
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<tr>
<td>(B,H)</td>
<td>5,5*</td>
<td>1,0</td>
<td>5,5</td>
<td>1,0</td>
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</tbody>
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* Nash Equilibrium

**Extensive Form → Normal Form always exists.**

Redundancies are present, e.g. (3,8) entries

**Normal Form → Extensive Form does not always exist.**

**Theorem:** Every finite perfect information game in extensive form has a pure-strategy Nash equilibrium.

In games of perfect information, you know precisely where in the tree you are located.

A subgame of a game $G$ rooted at $h$ is the restriction of $G$ to the descendent of $h$.

The subgame perfect equilibrium (SPE) of a game $G$ are all strategy profiles $s$ such that for any subgame $G'$ of $G$, the restriction of $s$ to $G'$ is N.E. of $G'$.

The centipede game

\[ \begin{array}{cccccc}
D & D & D & D & D & (3,5) D-Down \\
(1,0) & (0,2) & (3,1) & (2,4) & (4,3) \\
\end{array} \]

The only SPE is for each player to always choose $D$.,
Imperfect information game

\((X, A, H, Z, \chi, P, v, u, I)\)

Perfect info game

\(I = (I_1, \ldots, I_n)\) where \(I_i = (I_{i,1}, \ldots, I_{i,k_i})\) is an equivalence relation on \(\{ h \in H : P(h) = i \}\) where \(\chi(h) = \chi(h')\) and \(P(h) = P(h')\) whenever \(E_{i,j}\)

for which \(h \in I_{i,j}\) and \(h' \in I_{i,j}\).

\[\text{e.g.} \]

Information sets = \(\{1_L, 1_b\}\)

At \(1_b\), 1 has actions \(L\) or \(R\) regardless of what 2 decides.

Any normal game \(\Rightarrow\) imperfect information game.

\[\text{e.g. P.D.} \Rightarrow\]

Behavioral Strategies = A player's probabilistic choice at each node that are independent of other nodes. These are separate from mixed strategies.

In games of perfect recall, behavioral & mixed strategies coincide.
Repeated Games

A stage game is played multiple times by the same set of players.

A way to capture repeated games finite times is to show it in an imperfect-information game in extensive form.

Payoffs accumulate over time.

A stationary strategy is to reproduce the same strategy profiles repeatedly. This would be memoryless.

In an infinite version, payoffs could be averaged into the limit:

$$\lim_{K \to \infty} \frac{1}{K} \sum_{i=1}^{K} r_i(j)$$

$r_i(j)$ is the payoff at round $j$ of the repeated sequence.

With future discount factor of $0 < \beta < 1$, rewards are

$$\sum_{J=1}^{K} \beta^J r_i(J)$$